- Definition; FCV's as mappings
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**Functions of a complex variable**

\[ z = x + iy \quad \text{complex variable, } x, y \in \mathbb{R} \]

\[ f(z) = u(x, y) + i\sigma(x, y) \quad u, \sigma \in \mathbb{R}. \] Complex numbers can be represented by points in a 2D plane; thus, function \( f \) maps the \((x, y)\)-plane into the \((u, \sigma)\)-plane.

\[ y \quad \overset{f}{\rightarrow} \quad \sigma \quad \overset{f(z)}{\rightarrow} \quad u \]

**Examples of mapping by FCV**

(a) \( f(z) = z^2 = (x + iy)^2 = u + i\sigma \Rightarrow u = x^2 - y^2, \sigma = 2xy \)

(b) \( f(z) = e^z = e^x \cos y + i e^x \sin y \)

Note: The property \( e^{z_1 + z_2} = e^{z_1} e^{z_2} \) does hold.

Note that 90°-corners of a contour in the \( z \)-plane are mapped onto 90°-corners in the \((u, \sigma)\)-plane. Preservation of angles is a property of so-called **conformal** mappings. Most of the FCV we study give conformal mapping at all points except some points or lines called singularities.
Multi-valued functions

Example 1: \( f(z) = \sqrt{z} \).

For real \( z \) we have two possible values \( \pm |z|^{\frac{1}{2}} \).

Similarly, for a complex \( z = |z| e^{i\Theta} \) we have two "branches":

\[
\begin{align*}
    f_1(z) &= |z|^{\frac{1}{2}} e^{i\Theta/2} \\
    f_2(z) &= -|z|^{\frac{1}{2}} e^{-i\Theta/2}
\end{align*}
\]

In this definition the half-line \((0, \infty)\) is special:

- slightly above \( \Theta = +0 \), slightly below \( \Theta = 2\pi - 0 \), i.e., \( \Theta \) and therefore \( f_1, f_2 \) are discontinuous. This is inconvenient, i.e., a single-valued function, but unavoidable: for \( f(z) = \sqrt{z} \) there is no way to choose a branch that is continuous in the entire complex plane. The solution is to introduce branch cuts: think about a plane with points along certain lines removed.

Example: branch cut along \((0, \infty)\).

Mappings:

\[
\begin{align*}
    \frac{\mathbb{C}}{0} \quad \xrightarrow{f_1} \quad \frac{\mathbb{C}}{0} \\
    f_1(0) \quad \xrightarrow{f_1} \quad f_1(0) \\
    \frac{\mathbb{C}}{A_2} \quad \xrightarrow{f_2} \quad \frac{\mathbb{C}}{A_2} \\
    f_2(A_2) \quad \xrightarrow{f_2} \quad f_2(A_2)
\end{align*}
\]

Analogy: International Date Line on the Globe

\( \text{Sept. 21} \quad \text{N} \quad \text{Sept. 22} \)

(upper half-plane)

Note that \( f_1(a_1) \neq f_1(a_2) \), discontinuous across the branch cut.

(lower half-plane)

at \( z = 0 \)

Note that 360°-corner in the plane will the branch cut is mapped onto a line, i.e., 180°-corner. Thus, at \( z = 0 \) the mapping is NOT conformal.

\( z = 0 \) is a singularity of \( f(z) \). Typically branch cuts emanate away from singularities.
"Walk-arounds"

The reason why branch cuts start at singular points is to ensure that walks around such a singularity in the complex plane is forbidden. Let's remove the cuts and see what such a walk does. Suppose we start at $z=1$ with $f_1$ branch, i.e., $f(1) = 1$.

Continuing counterclockwise along the circle, we have a choice between $+e^{i\theta/2}$ and $-e^{-i\theta/2}$, here $\theta_{2-1} = 0$.

To keep $f(\theta)$ continuous we have to always choose $+\epsilon$ and measure $\theta$ as shown. But when we arrive back to $z=1$ from below, we get $\lim_{z \to 1^-} f(z) = \lim_{\theta \to 2\pi} (e^{i\theta/2}) = -1$.

Thus, once the walk-around is complete, we somehow switched to the other branch. $f(\theta)$ is not a continuous single-valued function.

For path not encircling $z=0$ (see fig.)

this "branch-swapping" does not happen.

(because $\theta$ is continuous)

- Freedom of choosing the cuts.

However, they engender different behavior of $f(z)$.
Example 2: \( f(z) = \sqrt{z - 1} \). Now the singularity is at \( z = 1 \).
Possible choice of the branch cut is shown. Two branches are defined by
\[
\begin{align*}
L_2 & \quad f_{L_2} = \pm |z-1|^{1/2} \cdot e^{i\phi/2}, \quad 0 < \phi < 2\pi.
\end{align*}
\]

- **Definition:** Riemann sheet = (complex-plane) + (branch-cuts) + (choice of branch)

For the same choice of cuts, a multi-valued FCV has several branches \( \Rightarrow \) several Riemann sheets.

- **Riemann surface:** Riemann sheets appropriately glued together along the branch cuts. The shape of the R.S. for \( f(z) = \sqrt{z} \) is shown.

Example:

One loop around \( z=0 \) (ABCDEA) takes one to the 2nd Riemann sheet; the second loop (AFGHA) returns one back.
Example 3 \( f(z) = \sqrt{(z-a)(z-b)} \)

(a) Branch cut as shown. Additional convention: at \( z = x + i0 \), \( x > b \) (on top of the right br. cut) \( f(z) = \text{(real and positive).} \)

Q1: What is the asymptotic behavior of \( f \) at \( z \to \pm \infty \)?

A1: \( f(z) = \sqrt{(z-a)(z-b)} \approx \sqrt{z^2} = z \text{ or } -z \). Which one?

Use the fact that \( f \) is continuous and follow its evolution along a certain conveniently chosen path. We have

Since, crudely speaking,

\[
 f(z) = \sqrt{z-a} \cdot \sqrt{z-b}, \quad \text{we must have}
 \]

\[
 f(z) = \pm |z-a|^{1/2} |z-b|^{1/2} \ \text{e}^{i \phi/2} \ \text{e}^{i \beta/2}
 \]

At point I, \( \alpha, \beta \to 0 \), and so to have \( f > 0 \),

we need "+" sign.

At point F, \( \alpha, \beta \to \pi \Rightarrow f(z) \approx |z| \cdot \text{e}^{i \pi/2} = i |z| \approx z \).

Q2: What about \( f(z) \), \( z = x - i0 \) (point G)?

A2: Consider the path shown. Angle \( \alpha = \text{const} = 0 \). The phase accumulation is due to change in \( \beta \). It is equal to \( 2\pi \) and occurs along the small circle around \( z = b \).

Thus, \( \text{e}^{i \beta/2} \to \text{e}^{i(\beta+2\pi)/2} = \text{e}^{i \beta/2} \cdot (-1) \).

And so \( f(z) = - |z-a(z-b)|^{1/2} \). Again, we have a sign change because we encircled a singularity (in this case, \( z = b \)).

Q3: \( f(z) \) as \( z \to -\infty \)?

A3: Starting from G (where \( f \) is real and negative), we can arrive at negative Im-axis by \( (\pi/2) \)-turn \( \Rightarrow f(z) \sim -z \) at \( z \to -\infty \).
(6) \( f(z) = \sqrt[3]{z-a(z-b)} \). A different choice of the cut. Is it legitimate? It does not forbid encirclement \( z=b \), for example. What's important, however, is whenever one encircles \( z=b \), one simultaneously encircles \( z=a \). Eqn gives \((-1)\), and altogether the function is multiplied by \((-1) \times (-1) = 1\). It remains single-valued!

Yet \( f(z) \) is different from the one in case (a). Thus, \( f(z) \to \pm 2 \) as \( z \to -\infty \).

Example 4 \( f(z) = (z^2-a)^{1/3} \). Similar to above \( (a=-1, b=+1) \) but it's the cubic root now,

\[ \begin{array}{ccc}
-1 & 1 \\
-1 & 1
\end{array} \]

\[ \begin{array}{ccc}
-1 & 1 \\
\text{still works ;} & \\
\text{no walk-arounds possible}
\end{array} \]

\[ \begin{array}{ccc}
\text{is not a valid branch cut.} & 1 \\
\text{A single walk-around brings the factor}
\end{array} \]

\[ e^{\frac{i\pi}{3}} e^{\frac{2i\pi}{3}} = e^{\frac{4i\pi}{3}} \neq 1. \]

Example 5: \( f(z) = (z^4-a)^{1/4} \). \( f \) has four branch points: \( z=\pm 1, \pm i \).

Most traditional choice of the cuts

\[ \begin{array}{ccc}
\text{These work too.}
\end{array} \]

\[ \begin{array}{ccc}
\text{i} & \text{i} & \text{i}
\end{array} \]
• Back to Example 2 (a). Let \( a = -1, b = 1 \), then \( f(z) = \sqrt{z^2 - 1} \). What does \( f \) map the complex plane to?

- \( z \in \text{upper half-plane} \): \( 0 < \arg(z) < \pi \)
  \[ \Rightarrow 0 < \frac{\arg(z)}{2} < \pi \Rightarrow f(z) \in \text{upper half-plane} \]
- \( z \in \text{lower half-plane} \): \( f(z) \in \text{upper half-plane} \)
  So, \( f(z) \) never belongs to the lower \( \frac{1}{2}\text{-plane} \).

Another way to see that:

\[ f(z) = k \left( h \left( g(z) \right) \right), \text{ where} \]
\[ g(z) = z^2; \quad h(z) = z - 1, \quad k(z) = \sqrt{z}. \]

Problem (homework) \( f(z) = \sqrt{z^2 - 1} \), the branch cut as in Example 2(b): What is the complex plane mapped into?
Example 4 \( f(z) = \ln z \).

For \( z \in \mathbb{R}, z > 0 \), \( f(z) \) has a standard definition in real numbers: it is the real solution \( f_0 \) of equation \( \exp [f] = z \).

However, this equation has an infinite number of other solutions:

\[ f = f_0 + 2\pi i n, \quad n = \text{integer}. \]

(This is because \( \exp(f + 2\pi i) = \exp(f) \)).

Singular point: \( z = 0 \) (here \( f_0 \to \pm \infty \)).

The standard branch cut; the corresponding standard definition

\[ \ln z = \ln |z| + i\theta, \quad -\pi < \theta < \pi. \]

Check:

\[ \exp (\ln z) = |z| \cdot \exp(i\theta) \], consistent with the picture.

Q: What is \( \ln(-1) = ? \)

A: The question is ambiguous because \( z = -1 \) belongs to the branch cut. One should distinguish between

\[ \ln (-1+i0) = \pi \] and \[ \ln (-1-i0) = -\pi. \]
Example 5: \( f(z) = \ln \left( z + \sqrt{z^2 - 1} \right) \). The new element is that

\[ f(z) = h(g(z)), \text{ where functions } h(z) = \ln z \text{ and } g(z) = z + \sqrt{z^2 - 1} \]

are both multivalued. In this case, the choice of branch-cuts should take care of singularities of both functions.

Let's adopt the standard convention that \( f(z) \) is real at points just above the real-positive semi-axis, at large enough \( z \) (in this case, at \( z > 1 \)).

Consider two "natural" ways in which this can be achieved:

(a) \( \begin{array}{c|c}
E & B \\
\hline
D & A \\
\end{array} \)

(b) \( \begin{array}{c|c}
E & B \\
\hline
C & A \\
\end{array} \)

\[ g(x+iy) = x + \sqrt{x^2 - 1}, \quad x > 1 \] in both cases.

What does \( g(z) \) map the complex plane into?

Case (a): it is the upper \( \frac{1}{2} \)-plane

Case (b)

Note that \( g \neq 0 \) and moreover, encircling the point \( g = 0 \) (the branch point of \( h(g) \)) is not possible \( \Rightarrow \) this is a valid branch-cut scheme.

Case (b): in the the entire plane except the unit circle at the origin. Although \( g = 0 \) is again absent (i.e., for all \( z \), \( g(z) \neq 0 \)), in this case one can freely make a full circle around it \( \Rightarrow \) \( h(g) \) is multi-valued. Thus, case (b) is not a valid branch-cut scheme.

Note: Case (a) corresponds to the conventional definition of \( \text{arccosh}(z) \).

Indeed, if \( z = \cosh u \), then \( f = \ln (\cosh u + \sqrt{\cosh^2 u - 1}) = \ln (e^u) = u \).