1. Modified Bessel function has an integral representation

\[ K_\nu(x) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-x \cos t + \nu t} dt. \]

Use the saddle-point method to show that for \( p \gg x \gg 1 \)

\[ K_{ip}(x) \sim \sqrt{2\pi} \left( p^2 - x^2 \right)^{-\frac{1}{4}} e^{-\frac{\pi}{2}p} \sin \phi, \]

where

\[ \phi = p \ln \left( \frac{p}{x} + \sqrt{\frac{p^2}{x^2} - 1} \right) - \sqrt{p^2 - x^2} + \frac{\pi}{4}. \]

Hint: the constant-phase contours of the integrand have the following topology in the 1st quadrant:

Explain in detail how the integration contour is constructed in your calculation.
2. Consider $I(n) = \int_0^\infty \cos(nt) \exp(ia \cos t) dt$. (a) Use integration by parts to show that for integer $n$, $I(n) \to 0$ as $n \to \infty$ faster than any power of $n$. What about non-integer $n$? (b) Use the steepest-descent method in the complex-$t$ plane to show that $I(n) \sim \sqrt{\pi / 2n} (iae / 2n)^n$ for integer $n$. **Hints:** Using symmetry arguments the integral can be extended to a larger interval where replacement $\cos(nt) \to \exp(int)$ does not change its value. After that end-point contours in the complex-$t$ plane can be added, similar to [Bender] Example 6.8 [Eq. (6.6.19)]. The contour can then be deformed into a constant-phase contour dominated by a single saddle-point.

3. One of the solutions of the ODE $x^{2a} y^{(n)}(x) = y$ is $y = x^{n-1} \exp(-1/x)$. (a) Verify this by expanding the exponential term in power series in $1/x$ and then differentiating the obtained expansion the necessary number of times. (b) Find the remaining $(n-1)$-linearly independent solutions. **Hint:** Examine how the solution of a closely related ODE $x^{2a} y^{(n)}(x) = ky$ scales with $k$ (where $k = \text{const}$) and use it as an insight.

4. Solve $y' = (y/x) + (1/y)$. **Hint:** Try $y^2 = u$ substitution.

5. Solve $x^2 y'' - 4xy' + 6y = x^4 \sin x$. **Hint:** The homogeneous version of this ODE is of Euler type. Once you have found its two solutions, solve the inhomogeneous ODE by the variation-of-parameters method.

6. (a) Solve $y' = x^2 + 2xy + y^2$. **Hint:** Use a common substitution method. (b) Solve (reduce to a transcendental equation) the ODE $y' = xy/(x^2 + y^2)$. **Hint:** Use another common substitution.

7. Solve $y' = (1 + x)y^2 / x^3$.

8. Solve a Riccati equation $y' + y^2 + (\sin 2x)y = \cos 2x$. **Hint:** Transform it to a 2nd-order linear ODE. Seek a trial solution of the obtained ODE in the form of $\cos x$ or maybe, $\sin x$. Once you succeeded, use the “factoring of the known solution” method to get the other linearly independent solution (it will not be an elementary function but instead a certain integral over a combination of elementary functions - it is acceptable for our purposes). Write the general solution of the 2nd-order ODE in terms of two arbitrary constants of integration, and then return back to the original dependent variable $y(x)$.

9. Solve $yy'' = 2(y')^2$. **Hint:** This ODE is autonomous.