Neumann Green's function

As discussed at lecture, the Green's theorem applied to $\psi(r)$ – the solution of $\nabla^2 \psi(r) = p(r)$, $r \in V$ and a suitable Green's function $G(r, r_0)$ that satisfies $\nabla_0^2 G(r_0, r) = \delta(r-r_0)$ yields the formula

$$\psi(r) = \int d^3r_0 \ G(r_0, r) \ p(r_0) + \int_S d\mathbf{s}(r_0) \ [\psi(r_0) \ \frac{\partial G(r_0, r)}{\partial n(r_0)} - G(r_0, r) \ \frac{\partial \psi(r_0)}{\partial n(r_0)}] .$$

The Neumann boundary condition are $\frac{\partial \psi(r)}{\partial n} \bigg|_{r \in \partial V} = u_1(r)$. Thus, it is tempting to say that we can choose $\frac{\partial G(r_0, r)}{\partial n} \bigg|_{r \in \partial V} = 0$, in which case $\psi$ is expressed in terms of the known quantities $p$ and $u_1$:

$$\psi = \int d^3r_0 \ G \ p - \int_S d\mathbf{s} \ G \ u_1 .$$

However, a problem arises if $V$ is finite because $\int_S \frac{\partial G}{\partial n} \ d\mathbf{s} = \int d^3r \ \nabla^2 G = \int d^3r \ \delta(r-r_0) = 1$. So, $\frac{\partial G}{\partial n} = 0$ is not permitted. The simplest permitted boundary condition is $\frac{\partial G(r_0, r)}{\partial n(r_0)} = \frac{1}{A} \ \text{const}$, where $A = \int_S d\mathbf{s}$ is area of $S$, and $r_0 \in S$, $r \ not \ \in V$. In this case

$$\psi(r) = \int d^3r_0 \ G(r_0, r) \ p(r_0) + C - \int_S d\mathbf{s}(r_0) \ G(r_0, r) \ u_1(r_0) , \ \ C = \frac{\int_S d\mathbf{s} \ \psi}{\int_S d\mathbf{s}} = \langle \psi \rangle = \text{const} .$$

(Thus, the solution is defined up to an arbitrary constant, as could be expected.)

Related issue: $u_1(r)$ must satisfy the self-consistency condition

$$\int_S d\mathbf{s} \ u_1(r) = \int_S d\mathbf{s} \ \frac{\partial u_1}{\partial n} = \int d^3r \ \nabla^2 u_1 = \int d^3r \ p(r) , \ \text{for the solution to exist.}$$

Problem: Solve the 2D Laplace equation $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \psi(x, y) = 0$ in the circle $r^2 \leq a^2$ subject to the boundary condition $\frac{\partial \psi}{\partial n} = u_1(p)$ at $r = a$. Assume that the self-consistency condition $\int_0^a dp \ u_1(p) = 0$ is satisfied.

Hint: Apply the method of images, $G(r, r_0) = G_0(r, r_0) + 2 \ G_0(r, r_1)$, $G_0(r, r_0) = \frac{1}{2\pi} \ln \frac{|r-r_0|}{|r-r_1|}$ is the free-space Green's function, $u_i$ is arbitrary, $G_0$ is the image charge to be found, and $r_1 = r_0 \ (a^2/r_0^2)$ is its location (see fig.)
2. The half-space \( z > 0 \) is occupied by a liquid with the sound velocity \( c \). The disk-shaped region \( \sqrt{x^2+y^2} < a \) of the surface is made to undergo oscillations of small velocity \( \vec{v} = v_x \hat{x} \) \( e^{-i\omega t} \). The remainder of the surface is kept at zero normal velocity, \( v_z = 0 \). Assume that the flow is irrotational, i.e., that there exists \( \psi(\vec{r}) \) such that \( \vec{v}(\vec{r}) = \nabla \psi \). The potential function \( \psi \) satisfies the usual wave equation in the bulk:

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi(\vec{r}, t) = 0.
\]

(a) Find the suitable Green's function by method of images and express \( \psi(\vec{r}, t) \) in the form of a definite integral.

(b) Show that \( \psi(x = y = 0, z, t) \sim \frac{1}{2} \frac{\omega a^2}{c} e^{i k z - i \omega t} \) at \( z \to a \).

3. A wire of radius \( b \) is immersed in an oil bath of infinite volume. The heat diffusion coefficient of both oil and the wire is equal to \( D \). Both are originally at temperature \( T_0 \). The wire is instantaneously heated up to temperature \( T_1 > T_0 \) by a current pulse. Show that the temperature a distance \( r \) from the wire axis at a time \( t \) is

\[
T = T_0 + \frac{T_1 - T_0}{2D} e^{-r^2/4Dt} \int_0^{2\pi} \int_0^b e^{-p^2/4Dt} I_0\left( \frac{rp}{2Dt} \right) \, dp \, d\theta,
\]

where \( I_0(z) = \int_0^{2\pi} \frac{1}{2\pi} \exp(iz\cos \theta) \, d\theta \) is the modified Bessel function.

Additionally, find the leading-order behavior of \( \Delta T(r) = T(r) - T_0 \) at short times, \( t \ll \frac{rb}{D} \) and large distances \( r \gg b \). Use the asymptotic formula \( I_0(z) \sim \frac{1}{\sqrt{2\pi z}} e^{z} \), valid for \( z > 0, z \gg 1 \).