1. Consider the eigenvalue problem \( \frac{d}{dx} (1-x^2) \frac{dy}{dx} + \lambda y = 0 \) on the interval \(-1 < x < 1\). According to Sturm-Liouville theory, for each \( \lambda \) we can have only one eigenfunction. Yet both \( y_1(x) = 1 \) and \( y_2(x) = \ln \left( \frac{1-x}{1+x} \right) \) satisfy the above equation with the same \( \lambda = 0 \). What is wrong with this argument?

2. Prove the result for the Green's function of the S-L problem given at lecture:

\[
\phi'' + (a^2 + \lambda) \phi = 0, \quad \phi(0) = \phi(1) = 0.
\]

\[
G(x, y) = -\sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2 - a^2} \sin(\pi nx) \sin(\pi ny)
\]

\[
= -\frac{1}{a \sin a} \left\{ \begin{array}{ll}
\sin ax \cdot \sin a(1-y), & 0 \leq x \leq y \leq 1, \\
\sin ay \cdot \sin a(1-x), & 0 \leq y \leq x \leq 1.
\end{array} \right.
\]

3. Find the normalized (with unit weight) eigenfunctions \( y_n(x) \) and the eigenvalues \( \lambda_n \) of the S-L problem \( \hat{L} y = x^2 y'' + 2xy' + \frac{1}{4} y = -\lambda y \), \( y(0) = y(1) = 0 \).

Find the coefficients \( a_n \) in the series solution \( y(x) = \sum_n a_n y_n(x) \) of the equation \( \hat{L} y(x) = \frac{1}{\sqrt{x}} \).