Consider the effect of external electric field \( E = E_0 e^{-i\omega t} + \text{c.c.} \) on an atom that contains one electron in a ground-state orbital \( \phi_0(x) \). We will assume that this electron can be excited to a higher orbital \( \phi_j(x) \) and neglect any other transitions. The time-dependent electron wavefunction is

\[
y(x,t) = \phi_0 e^{-\frac{e}{\hbar} E_0 t} + c_j(t) \phi_j e^{-\frac{e}{\hbar} E_j t},
\]

which satisfies the Schrödinger equation

\[
\hbar \frac{d}{dt} y = \left[ \hat{H}_0 - (e) E x \right] y(x,t).
\]

The resultant equation for \( c_j(t) \) is

\[
\hbar \frac{dc_j}{dt} = -(-e) \langle j|x|0 \rangle E(t) e^{i\phi_j t}, \quad \phi_j = \frac{E_j - E_0}{\hbar}.
\]

To derive this equation we assumed that \( c_j \) is small and discarded terms \( O(c_j^2) \). Next, as usual in the time-dependent perturbation theory, we assume that the electric field was turned on adiabatically slow in the distant past, i.e., that the actual \( t \)-dependence of \( E \) is \( E(t) = E_0 e^{-i\omega t} + \text{c.c.} \), where \( \Delta \) is an infinitesimally small positive quantity (so that \( E(t = -\infty) = 0 \)). Using the boundary condition \( c_j(-\infty) = 0 \), i.e., that the atom was in the ground state before the perturbation was turned on, we get:

\[
c_j(t) = -\frac{eE_0}{\hbar} \langle j|x|0 \rangle \left[ e^{i\omega \Delta t} - e^{-i\omega \Delta t} \right] e^{i\omega t}.
\]
Now we can calculate the dipole moment, \( p = \langle \gamma | (-e) x | \ell \rangle \).

We have:

\[
p = \langle 0 | (-e) x | 0 \rangle - e c j \ e^{i \delta t} \langle j | x | 0 \rangle - e c j \ e^{-i \delta t} \langle 0 | x | j \rangle + O(c^2).
\]

We will assume that the first term vanishes (no dipole moment in the ground state). We can safely set \( \lambda \rightarrow 0 \) and obtain the following leading-order expression for \( p \):

\[
p(t) = -\frac{e^2 \varepsilon_0}{h} |\langle j | x | 0 \rangle|^2 \left[ \frac{e^{i \omega t}}{\omega - \mathcal{N}_j} - \frac{e^{-i \omega t}}{\omega + \mathcal{N}_j} \right] - \frac{e^2 \varepsilon_0}{h} |\langle j | x | 0 \rangle|^2 \left[ \frac{e^{-i \omega t}}{\omega - \mathcal{N}_j} - \frac{e^{i \omega t}}{\omega + \mathcal{N}_j} \right]
\]

\[
= -\frac{e^2 \varepsilon_0}{h} |\langle j | x | 0 \rangle|^2 e^{-i \omega t} \left( \frac{1}{\omega - \mathcal{N}_j} - \frac{1}{\omega + \mathcal{N}_j} \right) + c.c.
\]

\[
= \frac{2e^2}{h} |\langle j | x | 0 \rangle|^2 \frac{\mathcal{N}_j}{\mathcal{N}_j^2 - \omega^2}. \mathcal{E}(t).
\]

The ratio of \( p \) and \( \mathcal{E} \) is referred to as polarisability:

\[
\lambda = \frac{p}{\mathcal{E}} = \frac{2e^2}{h} |\langle j | x | 0 \rangle|^2 \frac{\mathcal{N}_j}{\mathcal{N}_j^2 - \omega^2}.
\]

It determines the dielectric function of an ensemble of atoms (or a weakly bound crystal of such atoms) according to the formula:

\[
\mathcal{E}(\omega) = 1 + 4\pi \eta_0 \lambda, \text{ where } \eta_0 = \text{atomic concentration}.
\]

This result can be rewritten in the form given in class:

\[
\mathcal{E}(\omega) = 1 + \frac{4\pi e^2 \eta_0}{m} \frac{S_j}{\mathcal{N}_j^2 - \omega^2}, \quad S_j = \frac{2m}{h} \mathcal{N}_j |\langle j | x | 0 \rangle|^2.
\]